Markov Chain Choice Model from Pairwise Comparisons

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***Abstract*—Recently, the Markov chain choice model has been introduced by Blanchet et al. to overcome the computational intractability for learning and revenue management for several modern choice models, including the mixed multinomial logit models. However, the known methods for learning the Markov models require almost all items to be offered in the learning stage, which is impractical. To address this challenge, we propose a new approach for learning the Markov chain models that only use pairwise comparisons. Thus learned Markov models provably enjoys the similar advantages of the original Markov chain choice models, such as recovering the multinomial logit model as a special case, approximation guarantees for mixed multinomial logit models, and tractable exact solutions to assortment optimization. We provide numerical simulations investigating the price we pay for the simplified learning approach, which is in the accuracy of the predicted probabilities.**

***Index Terms*—Choice Modeling, Assortment Optimization, Markov Chain Model.**

I. INTRODUCTION

A CENTRAL problem of interest to operations managers in many industries such as retailing and airlines is using collected historical sales data to predict the revenues or sales when offering a particular assortment of products. Such predictions are crucial in making important business decisions in assortment planning, new product development, brand value evaluation, demand estimation, optimal pricing, and revenue management. A classical example of such a decision problem is assortment planning, which aims to maximize the expected revenue by selecting the optimal subset of offerings or assortments under various constraints (e.g. spatial constraints, limited customer attention, etc.). Every revenue management problem essentially involves some variation of predicting customers' choices, and accurate prediction is crucial. *Discrete choice models* widely studied in marketing, economics, and psychology, provide natural tools for predicting customer behavior and solving decision problems.

Under a typical scenario, solving decision problems with choice models involve the following steps. First, the decision maker selects a choice model to be used based on her expertise and domain knowledge, and learns the model parameters that best explains the historical sales data. We refer to this step as solving the *learning problem*. Then, given the probabilistic model, the decision maker can use it to predict (or infer) the expected revenue, conditioned on a particular selection of offerings. In this step, choice models are used to answer questions such as, how many people will purchase this particular flight given that we offer these set of options. Next, given such predictions, the decision maker tries to maximize the revenue by solving the optimization problem, such as finding the optimal selection of assortments. For example, one might ask for the optimal set of flight schedules to offer, given that customers behave as modeled in previous steps. We refer to this step as solving the *decision problem*.

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Tradeoff between tractability and predictive power. The choice models applied to revenue management can be broadly classified into two classes, based on their tradeoff between complexity of the model and its predictive power. There are simple models described by a small number of parameters, usually linear in the number of products, including the Multi-Nomial Logit (MNL) model and some simple cases of the Random Utility Models (RUM). These simple models might be preferred in practice, since they are equipped with efficient solutions to learning and decision problems.

The MNL model is by far the most popular choice model studied in revenue management and has been applied to various real applications including marketing [?], [?] and travel demand modeling [?]. There are computationally and statistically efficient solvers for learning the MNL model. The maximum likelihood estimator for MNL models is a simple logit regression which can be solved very fast and very accurately from small number of historical data. For MNL models, predicting the revenue is trivial, and there are efficient solvers for the decision problem as well [?]. However, the tractability comes at the cost of limited *descriptive power* to represent complex distributions and limited *predictive power* to accurately predict the behavior of real customers. As a result, these simple models fail to capture the heterogeneity in substitution patterns across items which is common in real data.

On the other hand, there are complex choice models with strong descriptive and predictive powers, such as the Mixed MultiNomial Logit (MMNL) model, the Nested Logit (NL) model, and non-parametric models [?], [?]. For example, the MMNL model can approximate any choice model to arbitrary accuracy by mixing enough number of MNL models [?]. Given enough samples, any true distribution can be described accurately by MMNL, and thus predicted revenue and preferences will also be accurate. However, it is NP-hard to solve the assortment problem even with just two mixtures of MNL models [?]. Further, there is no computationally and statistically efficient methodology to learn MMNL models in general. Similar computational challenges are shared among all the complex models.

Search for tractable choice models with strong predictive powers. In this modern era of big data, the success of a retailer depends on the capability to efficiently harness the potential of enormous data generated from customers' activities. More than 30 million customers visit Walmart (online and offline) stores everyday, and more than 20 million customers visit Amazon.com everyday. To deal with such enormous data generated from their purchases and activities, computational efficiency is of primary interest.

Choice modeling was developed by economists and cognitive psychologists with the goal of providing a model for people's rational decision processes based on utility theory. This has led to RUMs and the MNL model, and numerous variations such as NL and MMNL. However, even with just two components, MMNL models are intractable, requiring exponential number of operations to solve the decision problems. Further, the predictability of such a model is very limited, since the number of parameters are only twice that of a simple MNL model. In terms of the tradeoff between computational efficiency and predictive power, this is significantly suboptimal. We are paying enormous computational cost for little gain in the predictive power. This sub-optimality in the tradeoff is due to the fact that these models were developed without computational cost in mind. Some of the recent approaches try to address this problem by proposing non-parametric models with strong predictive powers, but fail to provide tractable solutions to decision problems [**?**].

To fully harness the potential of such a big data, we are in a dire need for a choice model that is tractable and at the same time has a strong predictive and descriptive power. To this end, a radically different approach of embedding computational efficiency into the design of the choice models from the start has led to recent advances in *Markov chain choice models* [**?**]. A choice model provides a parametric probabilistic model that describes the probability that a customer purchases an item given a set of offered items., i.e.

$$\pi(i,S) \;=\; \mathbb{P}(\text{item } i \text{ purchased given a subset } S \text{ is offered})\;,$$

for all $i \in S$ and $S \subseteq \mathcal{N}$, where $\mathcal{N}$ is the set of all items. In the learning problem, the parameters of the choice model is learned from historical purchase data. Then, the following assortment optimization is solved in the decision problem stage:

$$\max_{S\subseteq\mathcal{N}} \sum_{i\in S} r_i\,\pi(i,S)\,, \tag{1}$$

where r_i is the revenue generated when item i is sold, and we focus on this canonical example of unconstrained revenue maximization.

The Markov chain choice model enjoys several advantages. First, unlike other complex choice models, solving the above assortment optimization problem is tractable; there exists a polynomial time algorithm (Algorithm 1) for finding the optimal solution [**?**]. Several variations of the assortment optimization with various constraints have also been studied in [**?**], [**?**].

Second, the Markov chain choice model recovers the canonical MNL model when the data is generated from the MNL model. This is a desirable property, since the MNL model is by far the most widely studied choice model, and several other popular choice models are generalizations of MNL. Hence, it serves as a baseline for studying more complex choice models.

Finally, the Markov chain choice model provides a good approximation of the complex MMNL model with provable approximation guarantees. This is a desirable property since it is well known that any choice model can be approximated by a MMNL with sufficient number of mixtures up to an arbitrarily close precision [**?**]. Further, this in turn can be translated into similar multiplicative approximation guarantee in the solution of the assortment optimization, as studied in [**?**], [**?**].

All these merits of the new model comes at a cost. The complexity of learning the parameters of the Markov chain choice model blows up. Only known method for learning the model requires almost all the items in $\mathcal{N}$ to be offered in the learning stage, which is impractical. The model cannot be learned from typical historical purchase data. In the following, we present in detail the Markov chain choice model as introduced in [**?**] in Section II, and discuss the strengths and weaknesses.

Contributions. To overcome this intractability in learning the Markov chain choice model, we propose a new approach to learn the model in Section III, and prove that it shares all the advantages of the original model while requiring only pairwise comparisons data. Namely, we prove in Theorem III.1 that under the MNL model the proposed approach recovers the MNL model. We prove in Theorem III.2 that under the Mixed MNL model we achieve a bounded multiplicative approximation guarantee. Finally, we prove in Theorem IV.2 that this translates into an approximate solution to the assortment optimization problem, with strong multiplicative approximation guarantees. We present numerical simulation results for comparing those two approaches of learning the Markov chain choice models, and illustrate how much we need to pay in accuracy for the gain in complexity in learning.

II. The Markov Chain Choice Model

This section presents the *Markov chain choice model* introduced in [**?**]. We first define the necessary notations. Let $\mathcal{N} \equiv \{1,2,\cdots,n\}$ denote the set of all products, 0 denote the "no purchase option", $S \subseteq \mathcal{N}$ denote the set of products offered to an arriving customer, $S_+ = \{S,0\}$, $\pi(j,S)$ denote the probability that a customer chooses product $j \in S_+$ when S is the offer set, and λ_i denote the choice probability of a customer selecting product i when the offer set is $\mathcal{N}$, i.e., $\lambda_i = \pi(i,\mathcal{N})$.

Recall that a choice model represents the distribution of a customer's choice given offerings, i.e. $\pi(i,S) = \mathbb{P}(i|S_+)$ for all $S \subseteq \mathcal{N}$, and $i \in S$. A Markov chain choice model assigns this set of probabilities according to the following random process. A Markov chain $\mathcal{M}$ is defined over $\mathcal{N}_+$ states, arrival probability of $\{\lambda_i\}_{i\in\mathcal{N}}$, and the transition probability $\{\rho_{ij}\}_{i,j\in\mathcal{N}_+}$. Given this triple $(\mathcal{N}_+,\{\lambda_i\}_{i\in\mathcal{N}},\{\rho_{ij}\}_{i,j\in\mathcal{N}_+})$, the customer choice probability is assigned as follows. Consider a case when a set S is offered. A new arriving customer (conceptually) arrives at a particular item according to the arrival probability λ_i. Note that $\sum_{i \in \mathcal{N}} \lambda_i = 1$ and $\lambda_i \geq 0$ for all i by definition, and no customer arrives at 0 which is no purchase option. If the customer's arriving item is offered in S, the item is purchased. Otherwise, the customer moves to a substitution product $j \in \mathcal{N}_+$ with probability ρ_{ij} and if that product is available the substitution product is purchased. Moving to the state 0 corresponds to leaving the system without buying any product. This Markov chain $\mathcal{M}$ specifies the probability of customers choices, which is precisely stated in the following.

Theorem II.1 (Theorem 2.1 from [**?**]). *Suppose the parameters for the Markov chain model are given by λ_i, ρ_{ij} for all $i, j \in \mathcal{N}_+$. For any $S \subseteq \mathcal{N}$, let $\mathbf{B} = \boldsymbol{\rho}(\bar{S}, S_+)$ denote the transition probability sub-matrix from states $\bar{S} = \mathcal{N} \setminus S$ to S_+, and $\mathbf{C} = \rho(\bar{S}, \bar{S})$ denote the transition sub-matrix from states in $\bar{S}$ to $\bar{S}$. Then for any $i \in S_+$,*

$$\pi(i, S) = \lambda_i + (\boldsymbol{\lambda}_{\bar{S}})^\top (\mathbf{I} - \mathbf{C})^{-1} \mathbf{B} \mathbf{e}_i \,, \tag{2}$$

where $\boldsymbol{\lambda}_{\bar{S}}$ is the vector of arrival probabilities in $\bar{S}$ and $\mathbf{e}_i$ is the i-th unit vector.

The underlying assumption here is that the transition probability sub-matrix $\boldsymbol{\rho}(\mathcal{N}, \mathcal{N})$ is irreducible and has spectral radius strictly less than one. This assumption is fairly general. A transition matrix is irreducible if there is non-zero probability of to reach state i starting from state j for all $i, j \in \mathcal{N}$. Moreover, if we assume for any state $i \in \mathcal{N}$ there is non-zero probability of transitioning to state 0 that is $\rho_{i0} > 0$, then the spectral radius of $\boldsymbol{\rho}(\mathcal{N}, \mathcal{N})$ is strictly less than one.

This is the probability that the Markov chain starting with λ ends up at one of the nodes in the absorbing set S_+. The precise computation involves defining a new Markov chain $\mathcal{M}(S)$ over the state space $\mathcal{N}_+$ by setting the states corresponding to the products in set S_+ as absorbing states. Concretely, we define the transition matrix $\boldsymbol{\rho}(S)$ using $\boldsymbol{\rho}$ as follows:

$$\rho_{ij}(S) = \begin{cases} 1 & \text{if } i \in S_+, i = j \\ 0 & \text{if } i \in S_+, i \neq j \\ \rho_{ij} & \text{otherwise,} \end{cases} \tag{3}$$

The choice probability $\hat{\pi}(i, S)$ can be computed as the probability of absorbing state i in $\mathcal{M}(S)$.

This *Markov chain choice model* is introduced in the celebrated work in [**?**] and has several significant merits over existing choice models: (a) assortment optimization in (1) can be solved in polynomial time; (b) is a natural generalization of the popular MNL model; (c) is a good approximation of the complex Mixed MNL model with provable approximation guarantees. In other words, Markov chain choice model is a natural generalization of MNL that is designed with emphasis on the computational aspect. This is a leap forward from conventional designs which focus on prediction power of real world datasets rather than computational efficiency. We detail each of these advantages in the following section, with emphasis on how our proposed approach also enjoys similar merits.

However, this advances come at a cost of learning complexity. The only known method for learning the parameters λ_i's and $\rho_{i,j}$'s of the model from historical purchase data requires almost the entire set of items to be offered. Precisely, suppose we are given an accurate estimates of $\pi(i, \mathcal{N})$ for all $i \in \mathcal{N}$ and $\pi(i, \mathcal{N} \backslash \{i\})$ for $i, j \in \mathcal{N}$ from historical purchase history. Note that this requires almost all items to be offered at this learning stage. Then it is known from [**?**] that the following provides one approach to learn the parameters of the Markov chain model. Setting

$$\lambda_i \;=\; \pi(i, \mathcal{N}) \;, \text{ and} \tag{4}$$

$$\rho_{i,j} \;=\; \begin{cases} 1 & \text{if } i = 0, j = 0, \\ \frac{\pi(j, \mathcal{N} \backslash \{i\}) - \pi(j, \mathcal{N})}{\pi(i, \mathcal{N})} & \text{if } i \in \mathcal{N}, j \in \mathcal{N}_+, i \neq j, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

provably learns the correct Markov chain model given we accurately learned $\pi(i, \mathcal{N})$'s and $\pi(j, \mathcal{N} \backslash \{i\})$'s. It is suggested that when this data is not available, one can replace the entire offering set $\mathcal{N}$ by a subset S of which the purchase history data is available, but without any guarantees. No other approach to learning this Markov chain model has been proposed, to the best of our knowledge.

III. Main Results

In this section, we propose a new approach to learn the Markov chain model when only restricted data is available, in the extreme case where only pairwise comparisons are available. This provides a flexible approach for learning, since pairwise relations can often be easily deduced from higher-order relations such as purchase history with any set of offerings. Suppose, for each pair of products, we have accurate estimates of $\pi(i, \{i, j\})$ and $\pi(j, \{i, j\})$. We replace the learning approach of (5) with the following, procedure and set

$$\rho_{ij} = \begin{cases} \gamma \dfrac{\pi(j, \{i, j\})}{\pi(i, \{i, j\})} & \text{if } i \neq j \\ 1 - \gamma \displaystyle\sum_{k \in \mathcal{N}_+ \backslash \{i\}} \dfrac{\pi(k, \{i, k\})}{\pi(i, \{i, k\})} & \text{if } i = j \,, \end{cases} \tag{6}$$

where $\gamma = \min_{i,j}\{\pi(i, \{i, j\})\}/(n-1)$. The choice of γ is not critical, and as long as it ensures that the probabilities are non-negative and sum to one.

As computing $\lambda_i = \pi(i, \mathcal{N})$ also requires the entire set to be offered, we use pairwise comparisons again. Concretely, we define for $i \in \mathcal{N}_+$

$$\lambda_i = \frac{\sqrt{\nu_i}}{\sum_{j \in \mathcal{N}_+} \sqrt{\nu_j}} \,, \tag{7}$$

where $\nu = \{\nu_1, \cdots, \nu_{n+1}\} \in \mathbb{R}_+^{n+1}$ is the unique stationary distribution of the Markov chain $\mathcal{M}$ with transition probabilities ρ_{ij} defined above in Equation (6). Note here that we assume the Markov chain is irreducible and aperiodic such that the stationary distribution is unique. The proposed approach to learn the Markov chain model is based on the MNL model, and enjoys the several benefits of the original learning approach in (4) and (5), as we detail below.

A. Exact Guarantees under the Multinomial Logit Model

The MNL model model is one of the first choice model to be studied and widely applied in real-world applications, and most of the choice models that follow are generalizations of this baseline model. Hence, a desirable property for any new choice model is that if the data comes from the MNL model then we recover the MNL model back. The Markov chain model with the original learning approach of (4) and (5) also enjoys this property [?]. In this section, we provide a brief background on the MNL model and prove that the proposed learning approach from pairwise comparisons also recovers the MNL model when the data is from MNL.

The MNL model is a special case of a general class of choice models referred to as *random utility models*, defined as follows [?], [?]. The true latent utility of an item i is parametrized by $u_i > 0$. When presented with a set of offerings, a user's reveled preference over those offerings is a ranked list, and it is sorted to noisy observation of the utilities, i.e. i.i.d. noise added to the true utility u_i's.

The MNL model is a special case where the noise follows the standard Gumbel distribution, and is one of the most popular model in choice theory [?], [?]. MNL has several important properties, making this model realistic in various domains, including marketing [?], transportation [?], [?], biology [?], and natural language processing [?]. The MNL model (a) satisfies 'independence of irrelevant alternatives' from social choice theory [?]; (b) has a maximum likelihood estimator that is a convex optimization in u; and (c) has a simple characterization of choice probabilities:

$$\pi_{\text{MNL}}(i,S) \quad = \quad \frac{w_i}{\sum_{j\in S_+} w_j}\,, \tag{8}$$

where $w_i = e^{u_i}$.

The Markov chain model, together with the learning approach of (6) and (7), takes as input the pairwise comparisons data on $\{\pi(i,\{i,j\})\}$'s and outputs the choice probabilities $\{\pi(i,S)\}$'s. The next theorem confirms that if the input is generated from MNL, the proposed approach successfully recovers the choice probabilities under the MNL model, which is a desired property of any choice mode. This implies that the MNL model is a special case of the proposed model and the learning approach.

Theorem III.1. *If the choice probabilities for pairwise comparisons $\pi(i,\{i,j\})$ for all $i,j\in\mathcal{N}_+$ are generated from a Multinomial Logit (MNL) model then for all $S\subseteq\mathcal{N}$, $i\in S_+$,*

$$\pi(i,S) = \pi_{\text{MNL}}(i,S)\,, \tag{9}$$

where $\pi(i,S)$ is the choice probability computed by the Markov chain choice model as defined in Section II and $\pi_{\text{MNL}}(i,S)$ is the true choice probability given by the underlying MNL model.

Proof. Suppose the parameters for the MNL model are $w = \{w_0, w_1\cdots, w_n\}$ where $\sum_{j\in\mathcal{N}_+} w_j = 1$. Since the MNL model is scale invariant, we can always scale w_i's such that they sum to one. The choice probability $\pi(j,S)$ for any offer set S and any $i\in S_+$ is given by (8). The transition probability matrix $\boldsymbol{\rho}$ for the Markov chain $\mathcal{M}$ as defined in (6) can be computed as follows:

$$\rho_{ij} = \begin{cases} \gamma\dfrac{w_j}{w_i} & \text{if } i\neq j \\ 1-\gamma\dfrac{1-w_i}{w_i} & \text{if } i=j\,. \end{cases} \tag{10}$$

First, we show that if the underlying model is MNL, $\lambda_i = \pi(i,\mathcal{N})$ for all $i\in\mathcal{N}_+$. Recall that an irreducible Markov chain that satisfies the *detailed balance* equation: there exists a $\mu\in\mathbb{R}_+^{n+1}$ such that $\mu_i\rho_{ij} = \mu_j\rho_{ji}$ for all i,j, has unique stationary distribution $\nu_i \propto \mu_i$. Observe that the Markov chain $\mathcal{M}$, (10) when the underlying model is MNL, satisfies detailed balance equation with $\mu_i = w_i^2$. Therefore, combining Equations (7) and (8) we have,

$$\lambda_i \;=\; w_i \;=\; \pi_{\text{MNL}}(i,\mathcal{N})\,. \tag{11}$$

Consider an alternate Markov chain $\widehat{\mathcal{M}}$ defined over the states $i\in\mathcal{N}_+$ with transition probabilities $\hat\rho_{ij} = w_j$ for all $i,j\in\mathcal{N}_+$. We claim that the limiting distribution of the two Markov chains $\mathcal{M}$ and $\widehat{\mathcal{M}}$ for any non-empty subset S of absorbing states, $S\subseteq\mathcal{N}_+$, for any given arrival probabilities $\{\lambda_i\}_{i\in\mathcal{N}_+}$ are same when $\mathcal{M}$ is irreducible. It is easy to see that when arrival probabilities are $\lambda_i = w_i$, limiting distribution for any subset S of absorbing states for the Markov chain $\widehat{\mathcal{M}}$ is proportional to the parameters $\{w_i\}_{i\in S}$. Therefore, for all $S\subseteq\mathcal{N}$ and all $i\in S_+$, we have

$$\pi(i,S) = \frac{w_i}{\sum_{j\in S_+} w_j} = \pi_{\text{MNL}}(i,S)\,. \tag{12}$$

To prove the above claim, see that the probability of state transition from i to j for all $i\neq j$ in $\widehat{\mathcal{M}}$ and $\mathcal{M}$, (10) when the underlying model is MNL, is same:

$$\sum_{k=0}^{\infty}(\rho_{ii})^k\rho_{ij} \;=\; \sum_{k=0}^{\infty}\left(1-\frac{\gamma(1-w_i)}{w_i}\right)^k\frac{\gamma w_j}{w_i} = \frac{w_j}{1-w_i} \tag{13}$$

$$\sum_{k=0}^{\infty}(\hat\rho_{ii})^k\hat\rho_{ij} \;=\; \sum_{k=0}^{\infty}(w_i)^k w_j = \frac{w_j}{1-w_i}\,. \tag{14}$$

□

B. Approximation Guarantees under the Mixture of MNL Model

The next desirable property of a choice model is that it provides an accurate approximations to the Mixed MNL (MMNL) models. The Markov chain model with the original learning approach of (4) and (5) also enjoys this property with provable approximation guarantees [?]. We provide a similar provable approximation guarantees in Theorem III.2, and we defer the precise comparisons of the guarantees with those of [?] until later, when we have defined all necessary terminologies.

To understand how well our Markov chain model approximates any general choice model, we compute multiplicative lower and upper bounds on the choice probability $\pi(i,S)$ for any $S\subseteq\mathcal{N}$ and any $i\in S_+$ relative to the true choice probabilities $\pi_{\text{MMNL}}(i,S)$ when the underlying choice model is a mixture of multinomial logit models. Since for any random utility based discrete choice model there exists an MMNL model that approximates all choice probabilities within an arbitrary small additive error of $\epsilon > 0$ [**?**], it suffices to show bounds relative to the MMNL model. Although in our model, we do not need to have $\pi_{\text{MMNL}}(i, \mathcal{N})$ and we estimate λ_i from the stationary distribution of the Markov chain $\mathcal{M}$. However, for the purpose of the analysis under MMNL model, we assume that we have the true arrival probabilities that is $\lambda_i = \pi_{\text{MMNL}}(i, \mathcal{N})$ and analyze how the transition probabilities ρ_{ij}'s learned from pairwise comparisons perform. Instead, if we used the stationary distribution to learn λ also, then we get an additive guarantee on the λ, that results in additive approximation guarantee in the choice probabilities $\pi(i, S)$. For a strong multiplicative guarantee in Theorem III.2, this assumption on the arrival probability is necessary.

Suppose the underlying model is a mixture of K multinomial logit models. Let θ_k denote the probability that an arriving customer belongs to segment k, such that $\sum_{k\in[K]} \theta_k = 1$. Let $\{w_{ik}\}_{i\in\mathcal{N}_+, k\in[K]}$ denote the utility parameters for each segment k such that $\sum_{i=0}^{n} w_{ik} = 1$ and $w_{0k} > 0$, for all $k \in [K]$. To simplify notations, for any $S \subseteq \mathcal{N}_+$, let $w_k(S) \equiv \sum_{i\in S} w_{ik}$. Let $\bar{S} = \mathcal{N} \setminus S$. The choice probability $\pi_{\text{MMNL}}(j, S)$ for any offer set S and any $i \in S_+$, for the MMNL model is given by

$$\pi_{\text{MMNL}}(i, S) = \sum_{k\in[K]} \theta_k \frac{w_{ik}}{1 - w_k(\bar{S})} . \tag{15}$$

The transition probability matrix $\boldsymbol{\rho}$ for the Markov chain $\mathcal{M}$ as defined in (6) can be computed as follows when the underlying model is MMNL:

$$\rho_{ij} = \begin{cases} \gamma\Big(\sum_{k\in[K]} \frac{\theta_k w_{jk}}{w_{ik} + w_{jk}} \Big/ \sum_{k\in[K]} \frac{\theta_k w_{ik}}{w_{ik} + w_{jk}} \Big) & \text{if } i \neq j \\ 1 - \sum_{k\neq i} \rho_{ik} & \text{if } i = j \,. \end{cases} \tag{16}$$

With the assumption that $\lambda_i = \pi_{\text{MMNL}}(i, \mathcal{N})$, and by defining $\lambda(S)$ as following, we have

$$\lambda_i = \sum_{k\in[K]} \theta_k w_{ik} \;\text{ and }\; \lambda(S) \equiv \sum_{i\in S} \lambda_i = \sum_{k\in[K]} \theta_k w_k(S) \,. \tag{17}$$

For any $S \subseteq \mathcal{N}$, let

$$\alpha(S) \equiv \max_{k\in[K]} w_k(S) \,, \;\text{ and }\; \beta(S) \equiv \min_{k\in[K]} w_k(S) \tag{18}$$

that is $\alpha(S)$ and $\beta(S)$ are the maximum and minimum probabilities that the most preferable product for any customer across all the segments belongs to the set S. Note that by definition, $\beta(S) \leq \lambda(S) \leq \alpha(S)$. Also, for any set $S \subseteq \mathcal{N}$, define

$$\tau(S) \equiv \min\{\kappa | \boldsymbol{\nu} \leq \boldsymbol{\lambda}_S \leq \kappa\boldsymbol{\nu}, \boldsymbol{\nu} \text{ is a top left eigenvector of } \widetilde{\boldsymbol{\rho}}(S, S)\}, \tag{19}$$

where $\boldsymbol{\lambda}_S$ is the vector of arrival probabilities $\{\lambda_i\}_{i\in\mathcal{N}_+}$ restricted to the set S and $\widetilde{\boldsymbol{\rho}}(S, S)$ is the transition sub-matrix from states S to S of Markov chain $\widetilde{\mathcal{M}}$ constructed from Markov chain $\mathcal{M}$ such that the self loops are removed and the rows are normalized to make the sum equal to one. Precisely, the transition matrix $\widetilde{\boldsymbol{\rho}}$ of $\widetilde{\mathcal{M}}$ is given by

$$\widetilde{\rho}_{ij} = \begin{cases} \frac{\rho_{ij}}{\sum_{a\in\mathcal{N}_+\setminus\{i\}} \rho_{ia}} & \text{if } i \neq j \\ 0 & \text{if } i = j \,. \end{cases} \tag{20}$$

Note that $\widetilde{\boldsymbol{\rho}}(S, S)$ is irreducible and therefore from Perron-Frobenius theorem, the top eigenvector $\boldsymbol{\nu}$ has all the entries non-negative which implies that $\tau(S)$ is well defined. Define σ to be:

$$\sigma \;\equiv\; \min_{i,j\in\mathcal{N}_+, k\in[K]} \frac{w_{ik}}{w_{jk}} \,. \tag{21}$$

Therefore $\sigma \leq 1$ is the minimum ratio of choice probability of any two items across all the segments when all the products are offered. We prove upper and lower bounds on the relative error between choice probabilities computed from Markov chain model $\pi(i, S)$ and the true choice probabilities $\pi_{\text{MMNL}}(i, S)$ of the underlying MMNL model that depend upon σ, $\lambda(\bar{S})$, $\alpha(\bar{S})$, $\beta(\bar{S})$ and $\tau(\bar{S})$. In particular, we prove the following theorem. We assume that $\max_{i\in\mathcal{N}_+} \lambda_i \leq 1 - 1/C$ for some constant $C > 1$.

Theorem III.2. *Suppose the underlying choice model for the true choice probabilities $\pi_{\text{MMNL}}(i, S)$ is an MMNL model. If the arrival probabilities $\lambda_i = \pi_{\text{MMNL}}(i, \mathcal{N})$ for all $i \in \mathcal{N}_+$, for the Markov chain model, then for all $S \subseteq \mathcal{N}$ and any $i \in S_+$,*

$$(1 - \alpha(\bar{S}))(1 + \sigma^2\lambda(\bar{S})) \;\leq\; \frac{\pi(i, S)}{\pi_{\text{MMNL}}(i, S)} \;\leq\; (1 - \beta(\bar{S}))\Big(1 + \frac{C\tau(\bar{S})\lambda(\bar{S})}{\sigma^2 \min\{0, (1 - C\lambda(\bar{S}))\}}\Big) \,. \tag{22}$$

where $\hat{\pi}(i, S)$ is the choice probability computed computed by the Markov chain choice model as defined in Section II.

Note that the upper bound is at least one, which follows from the fact that $\beta(\bar{S}) \leq \lambda(\bar{S}) \leq \alpha(\bar{S})$. Also, the upper bound is meaningful when $\lambda(\bar{S}) \leq 1/C$. When the offer set S is large, then $\alpha(\bar{S})$ is typically small and we get a sharp lower bound on $\pi_{\text{MMNL}}(i, S)$. In [**?**], it was proven that with the original learning approach with the entire items in the offer set, one can get a tighter lower bound of order $(1 - \alpha(\bar{S})^2)$ as opposed to the above $(1 - \alpha(\bar{S}))$. Similarly, for the upper bound, with sufficiently large S, we have typically small $\lambda(\bar{S})$ and $\tau(\bar{S})$.

Proof of Theorem III.2. We first give the proof of lower bound.

Proof of lower bound. With Equations (16), (17) and (21), we get the following lower and upper bound on transition probabilities ρ_{ij}, for $i \neq j$,

$$\gamma\sigma\frac{\lambda_j}{\lambda_i} \leq \rho_{ij} \leq \gamma\frac{1}{\sigma}\frac{\lambda_j}{\lambda_i} \,. \tag{23}$$

From Section II, we have a lower bound on the choice probability $\pi(i,S)$ for any $S \subseteq \mathcal{N}$ and $i \in S_+$,

$$\begin{aligned}\pi(i,S) &\geq \lambda_i + \sum_{j\in\bar{S}} \lambda_j \frac{\rho_{ji}}{\sum_{a\in\mathcal{N}_+\setminus\{j\}} \rho_{ja}} && (24)\\ &\geq \lambda_i + \sigma^2 \sum_{j\in\bar{S}} \lambda_j \frac{\lambda_i}{1-\lambda_j} && (25)\\ &\geq \lambda_i + \sigma^2 \lambda_i \lambda(\bar{S}) && (26)\end{aligned}$$

where in (24), $\rho_{ji}/(\sum_{k\in\mathcal{N}_+\setminus\{j\}} \rho_{jk})$ is the probability of transitioning to state i in one out of state transition. (25) follows from (23), and (26) uses the fact that $\lambda_j < 1$ and the definition of $\lambda(S)$.

Combining Equations (15) and (26), we have

$$\begin{aligned}&\pi_{\mathrm{MMNL}}(i,S) - \pi(i,S)\\ &\leq \sum_{k\in[K]} \theta_k w_{ik}\left(\frac{1}{1-w_k(\bar{S})} - 1\right) - \sigma^2\lambda(\bar{S}) \sum_{k\in[K]} \theta_k w_{ik} && (27)\\ &\leq \alpha(\bar{S})\pi_{\mathrm{MMNL}}(i,S) - \sigma^2\lambda(\bar{S})(1-\alpha(\bar{S})) \sum_{k\in[K]} \frac{\theta_k w_{ik}}{1-w_k(\bar{S})} && (28)\\ &\leq \pi_{\mathrm{MMNL}}(i,S)\Big(\alpha(\bar{S}) - (1-\alpha(\bar{S}))\sigma^2\lambda(\bar{S})\Big), && (29)\end{aligned}$$

where (27) follows from (17). Equations (28) and (29) uses the definition of $\alpha(S)$, (18).

Proof of upper bound. To prove the upper bound we use the fact that the Markov chain $\mathcal{M}$ is equivalent to another Markov chain $\widetilde{\mathcal{M}}$ for computing limiting distribution for any subset $S \subseteq \mathcal{N}$ of absorbing states, where $\widetilde{\mathcal{M}}$, (20), is constructed from $\mathcal{M}$ such that the self loops are removed and the rows are normalized to make the sum equal to one. From Section II, we know that

$$\begin{aligned}\pi(i,S) &= \lambda_i + (\boldsymbol{\lambda}_{\bar{S}})^\top (\mathbf{I} - \widetilde{\mathbf{C}})^{-1} \widetilde{\mathbf{B}} \mathbf{e}_i\\ &= \lambda_i + \sum_{q=0}^{\infty} (\boldsymbol{\lambda}_{\bar{S}})^\top \widetilde{\mathbf{C}}^q \widetilde{\mathbf{B}} \mathbf{e}_i\,, && (30)\end{aligned}$$

where $\widetilde{\mathbf{C}}$ and $\widetilde{\mathbf{B}}$ are sub-matrices of the transition matrix $\widetilde{\boldsymbol{\rho}}$. $\widetilde{\mathbf{C}} = \widetilde{\boldsymbol{\rho}}(\bar{S},\bar{S})$ and $\widetilde{\mathbf{B}} = \widetilde{\boldsymbol{\rho}}(\bar{S},S_+)$. Since for all $i \in \mathcal{N}$, there exists a $k \in [K]$ such that w_{ik} is strictly positive, all off-diagonal entries of both matrices $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{C}}$ are strictly positive. From our definition of $\tau(S)$, we have

$$\boldsymbol{\nu} \leq \boldsymbol{\lambda}_{\bar{S}} \leq \tau(\bar{S})\boldsymbol{\nu} \qquad (31)$$

where $\boldsymbol{\nu}$ is the eigenvector of $\widetilde{\mathbf{C}}$ corresponding to the maximum eigenvalue. Let ω be the maximum eigenvalue of $\widetilde{\mathbf{C}}$. Since all row sums of $\widetilde{\mathbf{C}}$ are strictly less than one, $\omega < 1$. Let $\tau \equiv \tau(\bar{S})$. For any $q \in \mathbb{Z}_+$, we have

$$(\boldsymbol{\lambda}_{\bar{S}})^\top \widetilde{\mathbf{C}}^q \widetilde{\mathbf{B}} \mathbf{e}_i \leq \tau\omega^q \boldsymbol{\nu}^\top \widetilde{\mathbf{B}} \mathbf{e}_i \leq \tau\omega^q (\boldsymbol{\lambda}_{\bar{S}})^\top \widetilde{\mathbf{B}} \mathbf{e}_i. \qquad (32)$$

Therefore, we have

$$\begin{aligned}\pi(i,S) &= \lambda_i + \sum_{q=0}^{\infty} (\boldsymbol{\lambda}_{\bar{S}})^\top \widetilde{\mathbf{C}}^q \widetilde{\mathbf{B}} \mathbf{e}_i\\ &\leq \lambda_i + \sum_{q=0}^{\infty} \tau\omega^q (\boldsymbol{\lambda}_{\bar{S}})^\top \widetilde{\mathbf{B}} \mathbf{e}_i && (33)\\ &= \lambda_i + \frac{\tau}{1-\omega} (\boldsymbol{\lambda}_{\bar{S}})^\top \widetilde{\mathbf{B}} \mathbf{e}_i\\ &= \lambda_i + \frac{\tau}{1-\omega} \sum_{j\in\bar{S}} \lambda_j \frac{\rho_{ji}}{\sum_{a\neq j} \rho_{ja}} && (34)\\ &\leq \lambda_i + \frac{\tau}{1-\omega}\frac{1}{\sigma^2} \sum_{j\in\bar{S}} \lambda_j \frac{\lambda_i}{1-\lambda_j} && (35)\\ &\leq \lambda_i\left(1 + \frac{C\tau}{\sigma^2(1-\omega)}\lambda(\bar{S})\right), && (36)\end{aligned}$$

where (33) follows from (32) and (34) follows from our definition of $\widetilde{\mathcal{M}}$. (35) uses the lower and upper bounds on ρ_{ij} from (23). (36) follows from the definition of $\lambda(\bar{S})$ and the assumption that $\min_{j\in\mathcal{N}_+} 1-\lambda_j \geq 1/C$.

Combining Equations (15) and (36), we have

$$\begin{aligned}&\pi_{\mathrm{MMNL}}(i,S) - \pi(i,S)\\ &\geq \sum_{k\in[K]} \theta_k w_{ik}\left(\frac{1}{1-w_k(\bar{S})} - 1\right) - \frac{C\tau\lambda_i\lambda(\bar{S})}{\sigma^2(1-\omega)}\\ &\geq \beta(\bar{S})\pi_{\mathrm{MMNL}}(i,S) - \frac{C\tau\lambda(\bar{S})}{\sigma^2(1-\omega)} \sum_{k\in[K]} \theta_k w_{ik} && (37)\\ &\geq \pi_{\mathrm{MMNL}}(i,S)\left(\beta(\bar{S}) - \frac{C\tau\lambda(\bar{S})}{\sigma^2(1-\omega)}(1-\beta(\bar{S}))\right), && (38)\end{aligned}$$

where (37) and (38) uses the definition of $\beta(\bar{S})$, (18). To complete the proof, we need to show that $\omega \leq C\lambda(\bar{S})$. ω is the maximum eigenvalue of $\widetilde{\mathbf{C}}$ which is bounded by the maximum row sum of $\widetilde{\mathbf{C}}$. Maximum row sum of $\widetilde{\mathbf{C}}$ is bounded by

$$\begin{aligned}\max_{j\in\bar{S}} \sum_{i\in\bar{S}} \widetilde{\rho}_{ji} &= \max_{j\in\bar{S}} \sum_{i\in\bar{S}} \frac{\rho_{ji}}{\sum_{a\neq j} \rho_{ja}} && (39)\\ &\leq \max_{j\in\bar{S}} \sum_{i\in\bar{S}} \frac{\lambda_i}{1-\lambda_j} && (40)\\ &\leq C\lambda(\bar{S})\,, && (41)\end{aligned}$$

where (39) follows from the definition of Markov chain $\widetilde{\mathcal{M}}$ and (40) uses the bounds on ρ_{ij} from (23). (41) follows from the definition of $\lambda(\bar{S})$ and the assumption that $\min_{i\in\mathcal{N}_+} 1-\lambda_i \geq 1/C$. □

IV. Assortment Optimization

In this section, we consider the problem of finding optimal assortment: the offer set S that maximizes the total revenue. To define the problem formally, let r_i denote the revenue of product $i \in \mathcal{N}$. The goal is to select an offer set $S \subseteq \mathcal{N}$ such that the total expected revenue is maximized in (1). We present the result from [?] that the Markov chain model can solve the optimization in polynomial time, and show that, with the approximation guarantee from the previous section, learning the transition probabilities from pairwise comparisons also achieves a bounded approximation guarantee for assortment optimization in Theorem IV.2.

Although, we cannot write the choice probability $\pi(i,S)$ in simple form, it turns out that the optimal assortment problem is solvable in polynomial time. We present an algorithm for selecting optimal set, originally given by [?]. For all $i \in \mathcal{N}$ and $S \subseteq \mathcal{N}$, let $g_i(S)$ denote the expected revenue when the offer set is S, from a customer whose first preference is i. From the definition of $\pi(i,S)$ as a probability over a Markov chain, it follows that the optimal assortment problem, (1), can be reformulated as

$$\max_{S\subseteq\mathcal{N}} \sum_{i\in\mathcal{N}} \lambda_i \, g_i(S) \,. \tag{42}$$

The problem is related to the classical optimal stopping problem [?]. If product i is in the offer set S then the customer buys that product and the expected revenue $g_i(S) = r_i$. Otherwise, the customer makes substitution according to the transition probabilities of the Markov chain $\mathcal{M}$ and we get the following expected revenue:

$$g_i(S) = \sum_{j\in\mathcal{N}} \frac{\rho_{ij}}{\sum_{a\neq i}\rho_{ia}} g_j(S) = \sum_{j\in\mathcal{N}} \widetilde{\rho}_{ij} g_j(S) \,, \tag{43}$$

where the second equality follows from the definition of the transition matrix $\widetilde{\boldsymbol{\rho}}$, (20). Note that, since there are self-loops in Markov chain $\mathcal{M}$, probability of transition to state j from state i is given by $\rho_{ij}/\sum_{a\neq i}\rho_{ia}$.

Let g_i, for all $i\in\mathcal{N}$, denote the maximum expected revenue that can be obtained from a customer whose first preference is product i, where the maximization is taken over all possible offer sets S, i.e.,

$$g_i = \max_{S_i\subseteq\mathcal{N}} g_i(S_i) \,. \tag{44}$$

The above definition of g_i allows the optimal offer sets S_i to be different based on which product is the first preference of the customer. Given g_i's the following theorem characterizes the optimal offer set S.

Theorem IV.1 (Theorem 5.1 from [?])**.** *Let g_i for all $i\in\mathcal{N}$ be as defined in* (44)*. Let*

$$S = \{i \in \mathcal{N} \mid g_i = r_i\} \,. \tag{45}$$

Then S is optimal for the assortment optimization problem (42).

Intuitively, for any $i\in\mathcal{N}$, we have $g_i \geq r_i$, and if $g_i > r_i$, it's better not to offer the product i, as it would result in $g_i = r_i$. The surprising result is that the optimal offer set S_i's in (44) are same for all i which allows the optimal offer set to be characterized as given in the above theorem.

Further, g_i's can be computed using an iterative algorithm. Algorithm 1 computes g_i's in polynomial time, [?] Lemma 5.1, under the assumption that ρ_{i0} is polynomially bounded away from 0 for all $i\in\mathcal{N}$.

Algorithm 1 Iterative Algorithm to compute $\boldsymbol{g}$, [?]

Input: $\boldsymbol{\rho}$, $\{r_i\}_{i\in\mathcal{N}}$
Output: Estimate $\boldsymbol{g}$

```
Initialize g_i^(0) ← r_i for all i ∈ N, Δ ← 1, t ← 0
while Δ > 0 do
    t ← t + 1
    for all i = 1, 2, ··· , n do
        g_i^(t) ← max { r_i, Σ_{j≠i} ρ̃_ij g_j^(t−1) }
    end for
    Δ ← ||g^(t) − g^(t−1)||_∞
end while
```

A. Assortment Optimization for the MMNL model

In this section, we give an approximation bound on the optimal assortment revenue if the underlying random utility model is an MMNL model. If the Markov chain transition probabilities ρ_{ij} are computed using pairwise comparisons generated from an MMNL model, and the arrival probabilities are $\lambda_i = \pi_{\mathrm{MMNL}}(i,\mathcal{N})$, then we compute gap between the optimal assortment revenue of the Markov chain model and the optimal assortment revenue of the underlying MMNL model. In particular, we prove the following theorem.

Theorem IV.2. *Suppose the transition probabilities of the Markov chain model are computed from pairwise comparisons arising from an underlying MMNL model, $\pi(\cdot,\cdot)$, and the arrival probabilities $\lambda_i = \pi(i,\mathcal{N})$ for all $i\in\mathcal{N}_+$. Let S^*_{MMNL} be an optimal assortment for the MMNL model and S^* be an optimal assortment for the Markov chain model. Then*

$$\sum_{i\in S^*} r_i\pi(i,S^*) \;\geq\; \frac{\psi_1(\bar{S}^*_{\mathrm{MMNL}})}{\psi_2(\bar{S}^*)}\left(\sum_{i\in S^*_{\mathrm{MMNL}}} r_i\pi(i,S^*_{\mathrm{MMNL}})\right), \tag{46}$$

where

$$\begin{aligned}\psi_1(\bar{S}) &= (1-\alpha(\bar{S}))(1+\sigma^2\lambda(\bar{S}))\,,\\ \psi_2(\bar{S}) &= (1-\beta(\bar{S}))\left(1+\frac{C\tau(\bar{S})\lambda(\bar{S})}{\sigma^2\min\{0,(1-C\lambda(\bar{S}))\}}\right),\end{aligned} \tag{47}$$

where σ^2,$\alpha(\cdot)$, $\beta(\cdot)$, $\lambda(\cdot)$, $\tau(\cdot)$ and constant C are defined in III-B.

Proof of Theorem IV.2. From Theorem III.2, we know that for all $i\in S$,

$$\begin{aligned}\pi(i,S) &\leq \psi_2(\bar{S})\pi_{\mathrm{MMNL}}(i,S)\,,\\ \pi(i,S) &\geq \psi_1(\bar{S})\pi_{\mathrm{MMNL}}(i,S)\,.\end{aligned}$$

Then

$$\begin{aligned}&\sum_{i\in S^*} r_i\pi_{\mathrm{MMNL}}(i,S^*)\\ &\geq \frac{1}{\psi_2(\bar{S}^*)}\sum_{i\in S^*} r_i\pi(i,S^*)\\ &\geq \frac{1}{\psi_2(\bar{S}^*)}\sum_{i\in S^*_{\mathrm{MMNL}}} r_i\pi(i,S^*_{\mathrm{MMNL}})\\ &\geq \frac{\psi_1(\bar{S}^*_{\mathrm{MMNL}})}{\psi_2(\bar{S}^*)}\sum_{i\in S^*_{\mathrm{MMNL}}} r_i\pi_{\mathrm{MMNL}}(i,S^*_{\mathrm{MMNL}})\,,\end{aligned} \tag{48}$$

where (48) follows from the fact that S^* is the optimal offer set for the Markov chain model. □

V. Numerical Results

We perform a computational study on how well our Markov chain model estimates choice probabilities arising from a mixture of MNL models. We generate a random instance of an MMNL model and compute pairwise choice probabilities according to it. Using pairwise comparison probabilities, we compute transition probabilities ρ_{ij}'s of our Markov chain model as defined in (6), and arrival probabilities λ_i's from the stationary distribution of the Markov chain as given in (7). We then compare choice probabilities computed by our Markov chain model with the true choice probabilities given by the underlying MMNL model on randomly generated offer sets. We perform the same comparison for the Markov chain model given in [**?**]. They compute transition probabilities as $\rho_{ij} = (\pi(j, \mathcal{N} \setminus \{i\}) - \pi(j, \mathcal{N}))/\pi(i, \mathcal{N})$ and arrival probabilities as $\lambda_i = \pi(i, \mathcal{N})$. Note that the choice probabilities required to compute transition probabilities and arrival probabilities in our model and theirs model are entirely different. Our model needs only pairwise comparison probabilities while they need true arrival probabilities and choice probability of each product when every product is offered except one.

We follow the experimental setup used in [**?**]. To compare performance of the two Markov chain models with respect to the MMNL model, we generate L random offer sets of size between $n/3$ and $2n/3$. We consider different values of n between 10 and 1000, and number of segments $K = \lceil \log n \rceil$ and $L = 100$. For all assortments $S_1, \cdots S_L$, we compute maximum relative error in choice probability given by the Markov chain model with respect to the true choice probability of the underlying MMNL model. For any $S \subseteq \mathcal{N}$, $i \in S_+$, let π, $\pi^{\rm MC1}$ and $\pi^{\rm MC2}$ denote the choice probability of the MMNL model, our Markov chain model and the Markov chain model given in [**?**] respectively. For all $\ell \in [L]$, we compute

$$\begin{aligned} {\rm errMC1}(\ell) &\equiv 100 \max_{i \in S_\ell} \frac{|\pi^{\rm MC1}(i, S_\ell) - \pi(i, S_\ell)|}{\pi(i, S_\ell)}\,, \\ {\rm errMC2}(\ell) &\equiv 100 \max_{i \in S_\ell} \frac{|\pi^{\rm MC2}(i, S_\ell) - \pi(i, S_\ell)|}{\pi(i, S_\ell)}\,, \end{aligned}$$

to get the average maximum relative error as:

$$\begin{aligned} {\rm avg - errMC1} &\equiv \frac{1}{L}\sum_{\ell=1}^{L} {\rm errMC1}(\ell)\,, \\ {\rm avg - errMC2} &\equiv \frac{1}{L}\sum_{\ell=1}^{L} {\rm errMC2}(\ell)\,. \end{aligned}$$

We also compute the maximum relative error over all offer sets and all products as

$$\begin{aligned} {\rm max - errMC1} &= \max_{\ell \in [L]} {\rm errMC1}(\ell)\,, \\ {\rm max - errMC2} &= \max_{\ell \in [L]} {\rm errMC2}(\ell)\,. \end{aligned}$$

We generate a random instance of an MMNL model as follows. We generate utility parameters of each MNL model $\{w_{ik}\}_{i \in \mathcal{N}_+, k \in [K]}$ as i.i.d. samples of the uniform distribution in $[0, 1]$. For probability distribution over the arrival probability in each segment, we choose $\{\theta_k\}_{k \in [K]}$ in two ways - uniformly distributed and randomly distributed. For uniform distribution, we take $\theta_k = 1/K$ for all k, and for random distribution, θ_k is generated uniformly at random. Table I and Table II present the average and maximum relative error of the choice probabilities for the two Markov chain models. All the reported results are averaged over 100 instances. Both the Markov chain models perform similar on uniform mixture and random mixture of MNL models. Our model performs worse than the model given in [**?**] in both cases on both metrics: average relative error and maximum relative error. This is expected since we are only using pairwise comparisons, which is expected to perform worse. Higher order relations, such as those required to learn the Markov chain model are typically difficult to get hold of in practice, and this gap in the performance should be considered as the price we pay for using only simple pairwise comparisons.

Sewoong Oh Biography text here.

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List of Tables

TABLE I
RELATIVE ERROR OF OUR MARKOV CHAIN MODEL AND THE MARKOV CHAIN MODEL GIVEN IN [?] WITH RESPECT TO UNIFORM MIXTURE OF RANDOM MNL MODELS

n	K	MC2 (%)		MC1 (%)	
		avg	max	avg	max
10	3	3.04	33.03	14.85	57.04
20	3	2.95	20.64	20.93	67.50
30	4	2.61	17.84	21.33	53.70
40	4	2.41	16.95	22.22	48.58
50	4	2.28	15.04	23.58	59.91
60	5	2.02	10.20	21.57	39.33
80	5	1.84	11.38	22.91	45.49
100	5	1.72	7.24	24.08	41.25
150	6	1.39	6.58	22.95	38.72
200	6	1.27	5.62	24.01	42.39
500	7	0.85	3.52	25.34	42.70
1000	7	0.65	2.76	26.68	37.26

TABLE II
RELATIVE ERROR OF OUR MARKOV CHAIN MODEL AND THE MARKOV CHAIN MODEL GIVEN IN [?] WITH RESPECT TO RANDOM MIXTURE OF RANDOM MNL MODELS

n	K	MC2 (%)		MC2 (%)	
		avg	max	avg	max
10	3	2.84	46.34	14.02	68.07
20	3	2.98	51.77	20.57	66.57
30	4	2.57	18.42	21.64	52.10
40	4	2.45	16.04	24.27	52.44
50	4	2.37	15.18	25.97	59.67
60	5	2.07	11.91	24.29	50.12
80	5	1.91	12.62	25.87	49.99
100	5	1.78	18.48	26.72	47.41
150	6	1.50	9.50	26.42	41.73
200	6	1.32	7.19	26.88	43.13
500	7	0.91	4.46	28.14	46.17
1000	7	0.70	3.49	30.68	47.50